

# 一般容斥原理的数学归纳法证明

闫 浮

(北京化工大学信息科学与技术学院, 北京 100029)

摘 要: 文中对一般容斥原理的数学公式  $q_k^{(n)} = p_k^{(n)} - C_{k+1}^1 p_{k+1}^{(n)} + C_{k+2}^2 p_{k+2}^{(n)} - \dots \pm C_{n-k}^{n-k} p_n^{(n)} = \sum_{r=0}^{n-k} (-1)^r C_{k+r}^r p_{k+r}^{(n)}$

1)  $C_{k+1}^1 p_{k+1}^{(n)}$  进行了数学归纳法证明。

关键词: 一般容斥原理; 数学归纳法; 离散数学

中图分类号: O158

## 引 言

设有  $n$  个集合  $A_1, A_2, \dots, A_n$ , 笔者定义

$$p_k^{(n)} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} | \bigcap_{l=1}^k A_{i_l} |, \quad q_k^{(n)} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} | \overline{\bigcap_{j=1}^k A_{i_j}} |,$$

(其中  $k=1, 2, \dots, n$ )

$p_0^{(n)} = |E|$  (全集  $E$ ),  $q_0^{(n)} = | \overline{\bigcap_{j=1}^n A_j} |$ , 其中  $\overline{A_j}$

为  $A_j$  的补集。在一般容斥原理中当  $k=0$  时, 可表示如下

$$q_0^{(n)} = | \overline{\bigcap_{j=1}^n A_j} | = |E| - \sum_{j=1}^n |A_j| + \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| - \dots + (-1)^n | \bigcap_{r=1}^n A_r | = p_0^{(n)} - p_1^{(n)} + p_2^{(n)} - p_3^{(n)} + \dots + (-1)^n p_n^{(n)}$$

当  $k=n$  时, 可得到结论如下

$$p_n^{(n)} = | \bigcap_{r=1}^n A_r |, \quad q_n^{(n)} = | \overline{\bigcap_{r=1}^n A_r} |, \quad q_n^{(n)} = p_n^{(n)}$$

## 1 命题

$$q_k^{(n)} = p_k^{(n)} - C_{k+1}^1 p_{k+1}^{(n)} + C_{k+2}^2 p_{k+2}^{(n)} - \dots \pm$$

$C_{n-k}^{n-k} p_n^{(n)} = \sum_{r=0}^{n-k} (-1)^r C_{k+r}^r p_{k+r}^{(n)}$  称为一般容斥原理。

相对而言下式称为容斥原理或有穷集合的计数定理<sup>[1]</sup>。

$$\sum_{r=1}^n |A_r| = \sum_{i=1}^n |A_i| - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{n-1} | \bigcap_{r=1}^n A_r |$$

容斥原理证明比较简单<sup>[2]</sup>。本文证明一般容斥原理。

## 2 证明

为了证明方便引入记号

$$p_{k,n}^{(n-1)} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n-1} | \bigcap_{l=1}^k A_{i_l} \cap A_n |, \quad q_{k,n}^{(n-1)} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n-1} | \overline{\bigcap_{l=1}^k A_{i_l}} \cap A_n |$$

类似有

$$q_{k,n}^{(n-1)} = p_{k,n}^{(n-1)} - C_{k+1}^1 p_{k+1,n}^{(n-1)} + C_{k+2}^2 p_{k+2,n}^{(n-1)} - \dots \pm C_{n-1-k}^{n-1-k} p_{n-1,n}^{(n-1)} = \sum_{r=0}^{n-1-k} (-1)^r C_{k+r}^r p_{k+r,n}^{(n-1)}$$

相当于一般容斥原理中  $E = A_n$ , 而  $A_k$  相当于  $A_k \cap A_n$  ( $k=1, 2, \dots, n-1$ )

当  $n=1$  时,  $q_0^{(1)} = | \overline{A_1} | = |E| - |A_1| = p_0^{(1)} - p_1^{(1)}$ ,  $q_1^{(1)} = p_1^{(1)}$ , 命题成立。

设  $n=m$  命题成立, 即

$$q_k^{(m)} = \sum_{r=0}^{m-k} (-1)^r C_{k+r}^r p_{k+r}^{(m)} \quad (\text{其中 } k=0, 1, 2, \dots, m)$$

$q_{k,m+1}^{(m)} = \sum_{r=0}^{m-k} (-1)^r C_{k+r}^r p_{k+r,m+1}^{(m)}$  (其中  $k=0, 1, 2, \dots, m$ ) 则  $n=m+1$  时有

$$\begin{aligned}
 q_k^{(m+1)} &= \left| \begin{matrix} \overline{A_j} \\ \vdots \\ \overline{A_j} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_{k-1} \leq m+1} \left| \begin{matrix} A_{i_1} \\ \vdots \\ A_{i_k} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_k \leq m+1} \\
 &= \left| \begin{matrix} \overline{A_j} \\ \vdots \\ \overline{A_j} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_{k-1} \leq m} \left| \begin{matrix} A_{i_1} \\ \vdots \\ A_{i_k} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_k \leq m} \\
 &+ \left| \begin{matrix} \overline{A_j} \\ \vdots \\ \overline{A_j} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_{k-1} \leq m} \left| \begin{matrix} A_{i_1} \\ \vdots \\ A_{i_k} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_k \leq m+1} \\
 &= \left| \begin{matrix} \overline{A_j} \\ \vdots \\ \overline{A_j} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_{k-1} \leq m} \left| \begin{matrix} A_{i_1} \\ \vdots \\ A_{i_k} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_k \leq m} \\
 &- \left| \begin{matrix} \overline{A_j} \\ \vdots \\ \overline{A_j} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_{k-1} \leq m} \left| \begin{matrix} A_{i_1} \\ \vdots \\ A_{i_k} \end{matrix} \right|_{1 \leq i_1 < i_2 < \dots < i_k \leq m+1} \\
 &= q_k^{(m)} - q_{k, m+1}^{(m)} + \\
 &= \sum_{l=0}^{m-k} (-1)^l C_{k+l}^{(m)} p_{k+l}^{(m)} - \sum_{l=0}^{m-k} (-1)^l C_{k+l}^{(m)} p_{k+l, m+1}^{(m)} = \\
 &= p_k^{(m)} - C_{k+1}^1 p_{k+1}^{(m)} + C_{k+2}^2 p_{k+2}^{(m)} - C_{k+3}^3 p_{k+3}^{(m)} + \dots \pm
 \end{aligned}$$

$$\begin{aligned}
 &C_{m-k}^{m-k} p_m^{(m)} - p_{k, m+1}^{(m)} + C_{k+1}^1 p_{k+1, m+1}^{(m)} - C_{k+2}^2 \\
 &p_{k+2, m+1}^{(m)} + C_{k+3}^3 p_{k+3, m+1}^{(m)} - \dots \mp C_{m-k}^{m-k} p_{m, m+1}^{(m)} + \\
 &p_{k-1, m+1}^{(m)} - C_{k+1}^1 p_{k+1, m+1}^{(m)} + C_{k+2}^2 p_{k+2, m+1}^{(m)} - C_{k+3}^3 \\
 &p_{k+3, m+1}^{(m)} - \dots \mp C_{m-(k-1)}^{m-(k-1)} p_{m, m+1}^{(m)} = (p_k^{(m)} + \\
 &p_{k-1, m+1}^{(m)} - (C_{k+1}^1 p_{k+1}^{(m)} + C_{k+1}^1 p_{k+1, m+1}^{(m)}) + (C_{k+2}^2 p_{k+2}^{(m)} + \\
 &C_{k+2}^2 p_{k+2, m+1}^{(m)} + C_{k+1}^1 p_{k+1, m+1}^{(m)} - (C_{k+3}^3 p_{k+3}^{(m)} + \\
 &C_{k+3}^3 p_{k+3, m+1}^{(m)} + C_{k+2}^2 p_{k+2, m+1}^{(m)}) + \dots \pm \\
 &(C_{m-k}^{m-k} p_m^{(m)} + C_{m-1}^{m-1} p_{m-1, m+1}^{(m)} + C_{m-1-k}^{m-1-k} p_{m-1, m+1}^{(m)}) \mp \\
 &(C_{m-k}^{m-k} p_{m, m+1}^{(m)} + C_{m-k+1}^{m-k+1} p_{m, m+1}^{(m)} = p_k^{(m+1)} - C_{k+1}^1 \\
 &p_{k+1}^{(m)} + C_{k+2}^2 p_{k+2}^{(m+1)} - C_{k+3}^3 p_{k+3}^{(m+1)} + \dots \pm \\
 &C_{m-k}^{m-k} p_{m+1}^{(m+1)} \mp C_{m+1-k}^{m+1-k} p_{m+1}^{(m+1)}
 \end{aligned}$$

故  $n = m + 1$  命题成立, 类似可证关于  $q_{k, m+1}^{(m)}$  公式成立, 对一切  $n$ , 一切  $k$  一般容斥原理成立。

### 参 考 文 献

[1] 卢开澄. 组合数学. 北京: 清华大学出版社, 1999. 136 ~ 140  
 [2] 朱望规. 离散数学. 北京: 国防工业出版社, 1996. 10 ~ 11

## Inclusion and exclusion principle proved with mathematical induction

YAN Fu

(College of Information Science and Technology, Beijing University of Chemical Technology, Beijing 100029, China)

**Abstract:** This paper was concerned with proving general inclusion and exclusion principle which is quation:

$$q_k^{(n)} = p_k^{(n)} - C_{k+1}^1 p_{k+1}^{(n)} + C_{k+2}^2 p_{k+2}^{(n)} - \dots \pm C_n^{n-k} p_n^{(n)} = \sum_{l=0}^{n-k} (-1)^l C_{k+l}^{(n)} p_{k+l}^{(n)} \text{ in the way of mathematical induction.}$$

**Key words:** inclusion and exclusion principle; mathematical induction; discrete mathematics