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含有三阶色散和自频移与自陡峭项的立方-五次非线性薛定谔方程的孤子解

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摘 要: 研究了含有三阶色散、自频移与自陡峭项的立方-五次非线性薛定谔方程。根据齐次平衡原则, 运用 Riccati 方程-展开法、Riccati 方程-倒数展开法、Exp-展开法得到非线性薛定谔方程的几种精确解, 即亮-孤立波、激波、周期波, 图示了解的波形结构, 并比较了 3 种方法的关联性。

关键词: 非线性薛定谔方程; Riccati 方程-展开法; Exp-展开法; 齐次平衡原则; 孤立波

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引 言

非线性薛定谔方程广泛应用于非线性量子场论、等离子体物理学、光子学、流体力学、半导体电子学、生物物理学、经济物理学等诸多领域, 其表达式为

$$iu_t - \frac{\beta_2}{2}u_{xx} + \alpha_1|u|^2u = 0 \quad (1)$$

它是数学物理学中最重要的模型之一, 在 Bose-Einstein 凝聚中也被称为 Gross-Pitaevskii (GP) 方程。该方程存在孤立子解。由于孤立子在运动过程中有保持形状不变的性质, 可应用于光纤通信领域, 故寻找该系统的孤立子解具有重要意义。

在非线性光学中, 系统(1)描述了皮秒光脉冲在单模光纤中的传播^[1], 这里 $u(x, t)$ 表示复包络振幅, x 和 t 分别表示空间和时间变量, β_2 表示群速度色散系数, α_1 表示自相位调制参数。但是, 当超短脉冲接近 50 fs, 甚至低于 10 fs, 或光场频率接近光纤材料的谐振频率时, 单模光纤中高能超短脉冲传输

的理论和实验结果表明, 系统(1)失效, 这时必须考虑对系统(1)增加由拉曼散射引发的附加项^[2-3]。

文献[4]对系统(1)增加了五次非线性项、自陡峭项和自频移项

$$iu_t - \frac{\beta_2}{2}u_{xx} + (\alpha_1|u|^2 + \alpha_2|u|^4)u = i[\gamma_1(|u|^2u)_x + \gamma_2(|u|^2)_xu] \quad (2)$$

并且运用试探函数法得到孤子解。文献[5-8]对系统(1)增加了三阶色散项、自陡峭项和自频移项

$$iu_t - \frac{\beta_2}{2}u_{xx} + \alpha_1|u|^2u = i\left[\frac{\beta_3}{6}u_{xxx} + \gamma_1(|u|^2u)_x + \gamma_2(|u|^2)_xu\right] \quad (3)$$

同时运用了 Sine-Gordon 方程法、F-展开法、Exp-展开法等方法求解。

本文综合所有影响因素, 寻找系统

$$iu_t - \frac{\beta_2}{2}u_{xx} + (\alpha_1|u|^2 + \alpha_2|u|^4)u = i\left[\frac{\beta_3}{6}u_{xxx} + \gamma_1(|u|^2u)_x + \gamma_2(|u|^2)_xu\right] \quad (4)$$

的精确解。由于系统(4)中出现了五次非线性项 $\alpha_2|u|^4u$ 和三阶色散项 u_{xxx} 的并存现象, 导致一些常用的求解方法不适用, 为此我们选择合适的变量代换, 并选用 3 种辅助函数展开法—Riccati 方程-展开法、Riccati 方程-倒数展开法和 Exp-展开法来寻

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求方程(4)的解析解。

1 行波变换

对方程(4)作行波变换: $u(x, t) = \phi(\xi) e^{i\zeta}$, 其中 $\xi = v_0 t - vx, \zeta = \omega_0 t - \omega x$, 可得

$$\begin{cases} l_1 \phi''' + l_2 \phi' + l_3 \phi^2 \phi' = 0 \\ l_4 \phi'' + l_5 \phi + l_6 \phi^3 + \alpha_2 \phi^5 = 0 \end{cases} \quad (5)$$

式中, $l_1 = \frac{\beta_3}{6} v^3, l_2 = v_0 - \beta_2 \omega v - \frac{\beta_3}{2} \omega^2 v, l_3 = (3\gamma_1 +$

$2\gamma_2) v, l_4 = \left(-\frac{\beta_2}{2} - \frac{\beta_3}{2} \omega\right) v^2, l_5 = \frac{\beta_2}{2} \omega^2 + \frac{\beta_3}{6} \omega^3 - \omega_0,$

$l_6 = \alpha_1 - \gamma_1 \omega$ 。

对方程(5)第一个式子积分, 并选取首次积分常数为0, 通过代入消去 ϕ 的三次项, 可得

$$A\phi'' + B\phi + C\phi^5 = 0 \quad (6)$$

式中, $A = l_1 l_6 - l_3 l_4 / 3, B = l_2 l_6 - l_3 l_5 / 3, C = -\alpha_2 l_3 / 3$ 。

下面对方程(6)使用辅助函数展开法, 求解出函数 ϕ 的表达式, 从而给出方程(4)的解析解。

2 辅助函数展开法

设方程(6)的解具有如下展开结构

$$\phi = \sum_{k=0}^N A_k (f(V))^k, V'(\xi) = g(V(\xi)) \quad (7)$$

式中, A_k 为待定常数, N 为平衡整数。先将式(7)代入到方程(6)中, 再利用齐次平衡原理确定 N , 最后通过求解方程(7)给出系统(4)的解。为此选择代换 $\phi = Q^{1/4}$, 则方程(6)化简成

$$-3A(Q')^2 + 4AQQ'' + 16BQ^2 + 16CQ^3 = 0 \quad (8)$$

2.1 Riccati 方程-展开法

选取 $f(V) = V$ 和 $g(V) = S_2 V^2 + S_1 V + S_0$, 则式(7)变为

$$Q(\xi) = \sum_{k=0}^N A_k V^k(\xi), V'(\xi) = S_2 V^2(\xi) + S_1 V(\xi) + S_0 \quad (9)$$

式中, S_0, S_1 和 S_2 为常数且 $S_2 \neq 0$ 。易知 Riccati 方程的解如下。

当 $s > 0$ 时,

$$V(\xi) = -\frac{S_1}{2S_2} - \frac{\sqrt{s}}{2S_2} \tanh\left(\frac{\sqrt{s}}{2}(\xi + \xi_0)\right) \quad (10)$$

$$V(\xi) = -\frac{S_1}{2S_2} - \frac{\sqrt{s}}{2S_2} \coth\left(\frac{\sqrt{s}}{2}(\xi + \xi_0)\right) \quad (11)$$

当 $s < 0$ 时,

$$V(\xi) = -\frac{S_1}{2S_2} + \frac{\sqrt{-s}}{2S_2} \tan\left(\frac{\sqrt{-s}}{2}(\xi + \xi_0)\right) \quad (12)$$

$$V(\xi) = -\frac{S_1}{2S_2} - \frac{\sqrt{-s}}{2S_2} \cot\left(\frac{\sqrt{-s}}{2}(\xi + \xi_0)\right) \quad (13)$$

当 $s = 0$ 时,

$$V(\xi) = -\frac{S_1}{2S_2} - \frac{1}{S_2(\xi + \xi_0)} \quad (14)$$

式中, $s = S_1^2 - 4S_0S_2$, 且 ξ_0 为初始常数^[5]。

将方程(9)代入方程(8)后, 通过比较 Q^3 、 $(Q')^2$ 和 QQ'' 关于 V 的最高次幂, 可得 $N=2$ 以及关于 V 的一元高次方程。由于 $V \neq 0$, 于是选取 $V^k (k = 0, 1, \dots, 6)$ 的系数为0, 可得如下方程组

$$\begin{cases} V^6: 4(3AS_2^2 + 4A_2C)A_2^2 = 0 \\ V^5: 4(5AA_1S_2^2 + 4AA_2S_1S_2 + 12CA_1A_2)A_2 = 0 \\ V^4: (24A_0A_2 + 5A_1^2)AS_2^2 + 28A_1AA_2S_1S_2 + 8AA_2^2S_0S_2 + 4AA_2^2S_1^2 + 16(3A_0A_2C + 3A_1^2C + BA_2)A_2 = 0 \\ V^3: 8AA_0A_1S_2^2 + (40A_0A_2 + 6A_1^2)AS_1S_2 + 16AA_1A_2S_0S_2 + 8AA_1A_2S_1^2 + 16(6A_0A_2C + A_1^2C + 2BA_2)A_1 = 0 \\ V^2: 12AA_0A_1S_1S_2 + 2(16A_0A_2 + A_1^2)AS_0S_2 + (16A_0A_2 + A_1^2)AS_1^2 + 4AA_1A_2S_0S_1 - 4AA_2^2S_0^2 + 16(3A_0^2A_2C + 3A_0A_1^2C + 2BA_0A_2 + BA_1^2) = 0 \\ V: 8AA_0A_1S_0S_2 + 4AA_0A_1S_1^2 + 2(12A_0A_2 - A_1^2) \cdot AS_0S_1 - 4AA_1A_2S_0^2 + 16(3A_0C + 2B)A_0A_1 = 0 \\ V^0: 4A_1AA_0S_0S_1 + (8A_0A_2 - 3A_1^2)AS_0^2 + 16(A_0C + B)A_0^2 = 0 \end{cases}$$

借助 Maple 可求解出在 A_0 满足 $A(A_0C + 3B) <$

0 条件时, A_2, S_0, S_1, S_2 分别为 $A_2 = \frac{A_1^2C}{4(CA_0 + 3B)},$

$S_0 = \mp \frac{4A_0}{|A_1|} \sqrt{-\frac{A_0C + 3B}{3A}} \operatorname{sgn}(C), S_1 = \mp 4 \cdot$

$\sqrt{-\frac{A_0C + 3B}{3A}} \operatorname{sgn}(A_1C), S_2 = \frac{\pm |A_1C|}{\sqrt{-3A(A_0C + 3B)}} \cdot$

$\operatorname{sgn}(A)$, 其中 A_1 为任意非0常数, $s = -16B/A$ 。利用式(10)~(13)可得方程(4)的解如下。

当 $AB < 0$ 时,

$$u_1(x, t) = \left\{ -\frac{3B}{C} \operatorname{sech}^2\left(2\sqrt{-\frac{B}{A}}(v_0t - \right.\right.$$

$$vx) \left. \right\} \left. \right\}^{\frac{1}{4}} e^{i(\omega_0 t - \omega x)} \quad (15)$$

$$u_2(x, t) = \left\{ \frac{3B}{C} \operatorname{csch}^2 \left(2 \sqrt{-\frac{B}{A}} (v_0 t - vx) \right) \right\}^{\frac{1}{4}} \cdot e^{i(\omega_0 t - \omega x)} \quad (16)$$

当 $AB > 0$ 时,

$$u_3(x, t) = \left\{ -\frac{3B}{C} \sec^2 \left(2 \sqrt{\frac{B}{A}} (v_0 t - vx) \right) \right\}^{\frac{1}{4}} \cdot e^{i(\omega_0 t - \omega x)} \quad (17)$$

$$u_4(x, t) = \left\{ -\frac{3B}{C} \csc^2 \left(2 \sqrt{\frac{B}{A}} (v_0 t - vx) \right) \right\}^{\frac{1}{4}} \cdot e^{i(\omega_0 t - \omega x)} \quad (18)$$

选取参数 $\beta_2 = 1, \beta_3 = 1, \alpha_1 = 2, \gamma_1 = 1, \gamma_2 = 1, v_0 = 1, v = 1, \omega_0 = 1$ 进行图示。在 $\alpha_2 = -1, \omega = 2$ 时, 有 $A = \frac{5}{2}, B = -\frac{35}{9}, C = \frac{5}{3}$, 可得方程解 (15) 的波形

为亮-孤子(图 1(a)); 在 $\alpha_2 = 1, \omega = 2$ 时, 有 $A = \frac{5}{2},$

$B = -\frac{35}{9}, C = -\frac{5}{3}$, 可得方程解 (16) 的波形为激波

(图 1(b)); 在 $\alpha_2 = 1, \omega = 1$ 时, 有 $A = \frac{11}{6}, B = \frac{1}{18},$

$C = -\frac{5}{3}$, 可得方程解 (17)、(18) 的波形为周期波

(图 1(c)、(d)), 可以看出图 1(d) 在 $x = t$ 处出现了间断。

2.2 Riccati 方程-倒数展开法

选取 $f(V) = 1/V$ 和 $g(V) = S_2 V^2 + S_1 V + S_0$, 则式 (7) 变为

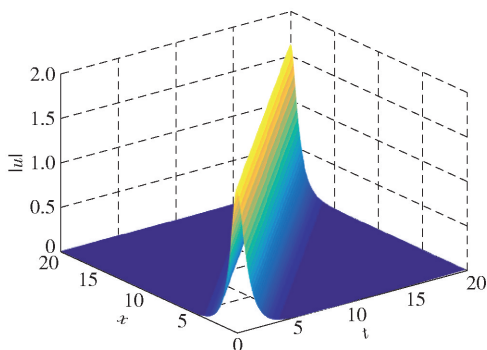
$$Q(\xi) = \sum_{k=0}^N A_k \left\{ \frac{1}{V(\xi)} \right\}^k, \quad V'(\xi) = S_2 V^2 + S_1 V + S_0 \quad (19)$$

将方程 (19) 代入方程 (8) 后, 通过比较 Q^3 、 $(Q')^2$ 和 QQ'' 关于 $1/V(\xi)$ 的最高次幂, 可得 $N=2$ 以及关于 $1/V(\xi)$ 的高次方程。由于 $V \neq 0$, 故选取 $(1/V(\xi))^k (k=0, 1, \dots, 6)$ 的系数为 0, 可得类似的系数方程组。借助 Maple 求解, 可知在 A_0 满足条件 $A(A_0 C + 3B) < 0$ 时, S_0, S_1, S_2, A_2 分别为

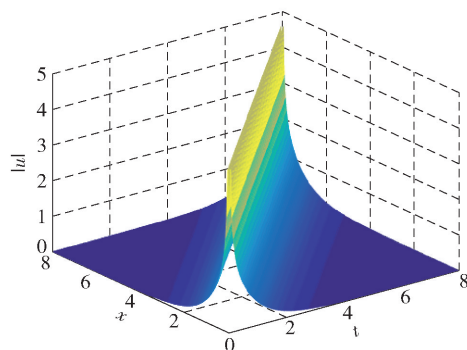
$$S_0 = \frac{\pm |A_1 C|}{\sqrt{-3A(A_0 C + 3B)}} \operatorname{sgn}(A)$$

$$S_1 = \mp 4 \sqrt{-\frac{A_0 C + 3B}{3A}} \operatorname{sgn}(A_1 C)$$

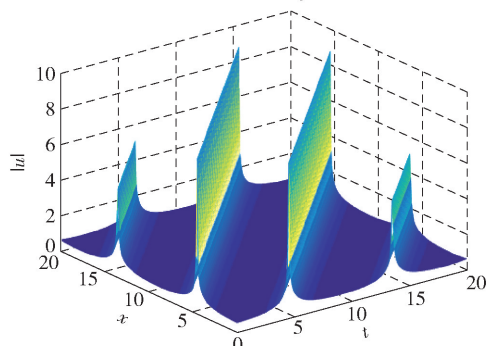
$$S_2 = \mp \frac{4A_0}{|A_1|} \sqrt{-\frac{A_0 C + 3B}{3A}} \operatorname{sgn}(C)$$



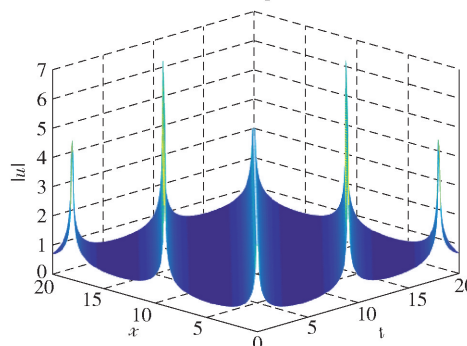
(a) 亮-孤立波 u_1



(b) 激波 u_2



(c) 周期波 u_3



(d) 周期波 u_4

图 1 亮-孤立波 u_1 , 激波 u_2 和周期波 u_3, u_4 的三维波形图

Fig. 1 Three-dimensional waveforms of the bright-soliton wave u_1 , shock wave u_2 and periodic waves u_3, u_4

$$A_2 = \frac{A_1^2 C}{4(CA_0 + 3B)}$$

当 $AB < 0$ 时,

$$u_5(x, t) = \left\{ \frac{-3A_0 B \operatorname{sech}^2 \left(2 \sqrt{-\frac{B}{A}} (v_0 t - vx) \right)}{(A_0 C + 3B) \left(\sqrt{\frac{3B}{A_0 C + 3B}} \tanh \left(2 \sqrt{-\frac{B}{A}} (v_0 t - vx) \right) - \operatorname{sgn}(A_1 C) \right)^2} \right\}^{\frac{1}{4}} e^{i(\omega_0 t - \omega x)} \quad (20)$$

$$u_6(x, t) = \left\{ \frac{3A_0 B \operatorname{csch}^2 \left(2 \sqrt{-\frac{B}{A}} (v_0 t - vx) \right)}{(A_0 C + 3B) \left(\sqrt{\frac{3B}{A_0 C + 3B}} \coth \left(2 \sqrt{-\frac{B}{A}} (v_0 t - vx) \right) - \operatorname{sgn}(A_1 C) \right)^2} \right\}^{\frac{1}{4}} e^{i(\omega_0 t - \omega x)} \quad (21)$$

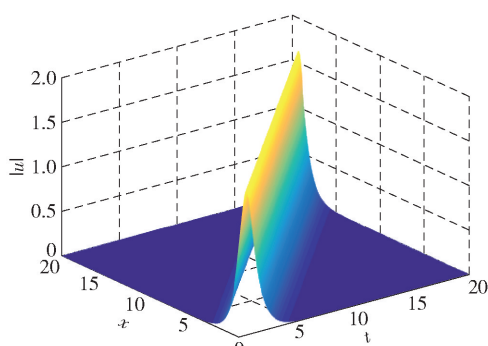
当 $AB > 0$ 时,

$$u_7(x, t) = \left\{ \frac{-3A_0 B \sec^2 \left(2 \sqrt{\frac{B}{A}} (v_0 t - vx) \right)}{(A_0 C + 3B) \left(\sqrt{-\frac{3B}{A_0 C + 3B}} \tan \left(2 \sqrt{\frac{B}{A}} (v_0 t - vx) \right) + \operatorname{sgn}(A_1 C) \right)^2} \right\}^{\frac{1}{4}} e^{i(\omega_0 t - \omega x)} \quad (22)$$

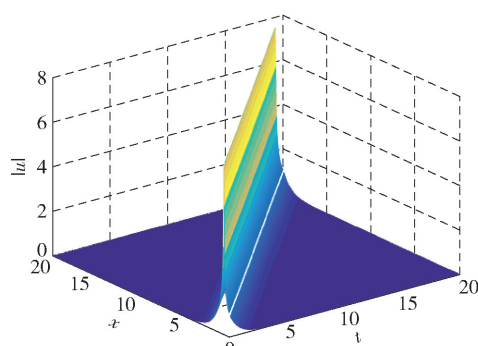
$$u_8(x, t) = \left\{ \frac{-3A_0 B \csc^2 \left(2 \sqrt{\frac{B}{A}} (v_0 t - vx) \right)}{(A_0 C + 3B) \left(\sqrt{-\frac{3B}{A_0 C + 3B}} \cot \left(2 \sqrt{\frac{B}{A}} (v_0 t - vx) \right) - \operatorname{sgn}(A_1 C) \right)^2} \right\}^{\frac{1}{4}} e^{i(\omega_0 t - \omega x)} \quad (23)$$

选取参数 $A_1 = 1, \beta_2 = 1, \beta_3 = 1, \alpha_1 = 2, \gamma_1 = 1, \gamma_2 = 1, v_0 = 1, v = 1, \omega_0 = 1$ 进行图示(图2)。当 $A_0 = -1$,

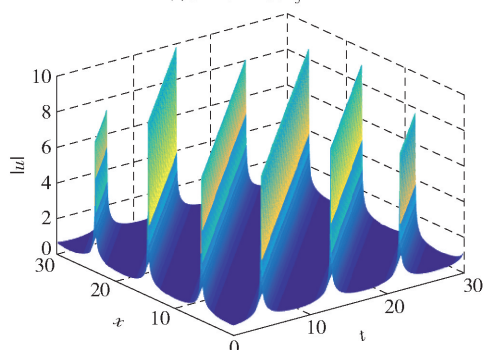
$\alpha_2 = -1, \omega = 2$ 时, 有 $A = \frac{5}{2}, B = -\frac{35}{9}, C = \frac{5}{3}$, 可得



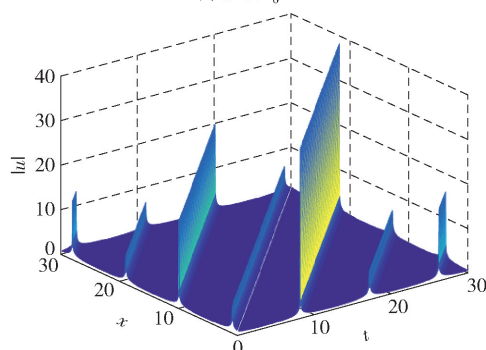
(a) 亮-孤立波 u_5



(b) 激波 u_6



(c) 周期波 u_7



(d) 周期波 u_8

图2 亮-孤立波 u_5 , 激波 u_6 和周期波 u_7, u_8 的三维波形图

Fig. 2 Three-dimensional waveforms of the bright-soliton u_5 , shock wave u_6 and periodic waves u_7, u_8

方程解(20)为亮-孤子(图2(a));当 $A_0 = 1, \alpha_2 = 1$,
 $\omega = 2$ 时,有 $A = \frac{5}{2}, B = -\frac{35}{9}, C = -\frac{5}{3}$,可得方程解
 (21)为激波(图2(b));当 $A_0 = 1, \alpha_2 = 1, \omega = 1$ 时,
 有 $A = \frac{11}{6}, B = \frac{1}{18}, C = -\frac{5}{3}$,相应可得方程解(22)、
 (23)的波形为周期波(图2(c)、(d)),可以看出图
 2(d)在 $x = t$ 处出现间断。

2.3 Exp-展开法

选取 $f(V) = \exp(-V)$ 和 $g(V) = \exp(-V) + R_1 \exp(V) + R_0$,则式(7)变为

$$Q(\xi) = \sum_{k=0}^N A_k \{ \exp(-V(\xi)) \}^k$$

$$V'(\xi) = \exp(-V(\xi)) + R_1 \exp(V(\xi)) + R_0 \quad (24)$$

式中 R_0 和 R_1 均为常数。易知上述关于 $V(\xi)$ 的微分方程的解如下。

当 $\mu > 0, R_1 \neq 0$ 时,

$$V(\xi) = \ln \left(-\frac{R_0}{2R_1} - \frac{\sqrt{\mu}}{2R_1} \tanh \left(\frac{\sqrt{\mu}}{2}(\xi + \xi_0) \right) \right) \quad (25)$$

$$V(\xi) = \ln \left(-\frac{R_0}{2R_1} - \frac{\sqrt{\mu}}{2R_1} \coth \left(\frac{\sqrt{\mu}}{2}(\xi + \xi_0) \right) \right) \quad (26)$$

当 $\mu < 0, R_1 \neq 0$ 时,

$$V(\xi) = \ln \left(-\frac{R_0}{2R_1} + \frac{\sqrt{-\mu}}{2R_1} \tan \left(\frac{\sqrt{-\mu}}{2}(\xi + \xi_0) \right) \right) \quad (27)$$

$$V(\xi) = \ln \left(-\frac{R_0}{2R_1} - \frac{\sqrt{-\mu}}{2R_1} \cot \left(\frac{\sqrt{-\mu}}{2}(\xi + \xi_0) \right) \right) \quad (28)$$

当 $\mu = 0, R_1 \neq 0, R_0 \neq 0$ 时,

$$V(\xi) = \ln \left(-\frac{2R_0(\xi + \xi_0) + 2}{R_0^2(\xi + \xi_0)} \right) \quad (29)$$

其中 $\mu = R_0^2 - 4R_1$,且 ξ_0 为初始常数^[5]。

将方程(24)代入方程(8)后,通过比较 Q^3 、 $(Q')^2$ 和 QQ'' 关于 $\exp(-V(\xi))$ 的最高次幂,可得 $N=2$ 以及关于 $\exp(-V(\xi))$ 的高次方程。由于 $\exp(-V(\xi)) \neq 0$,故选取 $\exp(-kV(\xi)) (k=0,1, \dots, 6)$ 的系数为0,可得如下方程组。

$$\begin{cases} \exp(-6V): 4(4CA_2 + 3A)A_2^2 = 0 \\ \exp(-5V): 4(4AA_2R_0 + 12CA_1A_2 + 5AA_1)A_2 = 0 \\ \exp(-4V): (48A_2C + 5A)A_1^2 + (4AR_0^2 + 8AR_1 + 16B)A_2^2 + 4(7AA_1R_0 + 6AA_0)A_2 = 0 \\ \exp(-3V): 8(AR_0^2 + 2AR_1 + 12A_0C + 4B)A_1A_2 + 40AA_0A_2R_0 + 2(3AA_1R_0 + 8CA_1^2 + 4AA_0)A_1 = 0 \\ \exp(-2V): (AR_0 - 4AR_1^2 + 2AR_1 + 48A_0C + 16B)A_1^2 + (16AR_0^2 + 32AR_1 + 48A_0C + 32B)A_0A_2 + 4(A_2R_1 + 3A_0)AA_1R_0 = 0 \\ \exp(-V): (4AR_0^2 + 8AR_1 + 48A_0C + 32B)A_0A_1 + (24A_0A_2 - 2A_1^2)AR_0R_1 - 4AA_1A_2R_1^2 = 0 \\ \text{Constant}: 4AA_0A_1R_0R_1 + (8A_0A_2 - 3A_1^2)AR_1^2 + 16A_0^2(A_0C + B) = 0 \end{cases}$$

借助 Maple 求解出 A_0, A_2, R_0, R_1 分别为

$$A_0 = -\frac{A_1^2 C^2 + 9AB}{3AC}, A_2 = -\frac{3A}{4C}$$

$$R_0 = -\frac{4A_1 C}{3A}, R_1 = \frac{4(A_1^2 C^2 + 9AB)}{9A^2}$$

其中 A_1 为任意常数,且 $\mu = R_0^2 - 4R_1 = -16B/A$ 。利用式(25)~(28),可得到方程(6)的解析解如下。

当 $AB < 0$ 时,

$$u_9(x, t) =$$

$$\left\{ \frac{-3(A_1^2 C^2 + 9AB)B \operatorname{sech}^2 \left(2\sqrt{-\frac{B}{A}}(v_0 t - vx) \right)}{C \left(3A\sqrt{-\frac{B}{A}} \tanh \left(2\sqrt{-\frac{B}{A}}(v_0 t - vx) \right) - A_1 C \right)^2} \right\}^{\frac{1}{4}} e^{i(\omega_0 t - ax)} \quad (30)$$

$$u_{10}(x, t) =$$

$$\left\{ \frac{3(A_1^2 C^2 + 9AB)B \operatorname{csch}^2 \left(2\sqrt{-\frac{B}{A}}(v_0 t - vx) \right)}{C \left(3A\sqrt{-\frac{B}{A}} \coth \left(2\sqrt{-\frac{B}{A}}(v_0 t - vx) \right) - A_1 C \right)^2} \right\}^{\frac{1}{4}} e^{i(\omega_0 t - ax)} \quad (31)$$

当 $AB > 0$ 时,

$$u_{11}(x, t) =$$

$$\left\{ -\frac{3(A_1^2 C^2 + 9AB)B \sec^2 \left(2\sqrt{\frac{B}{A}}(v_0 t - vx) \right)}{C \left(3A\sqrt{\frac{B}{A}} \tan \left(2\sqrt{\frac{B}{A}}(v_0 t - vx) \right) + A_1 C \right)^2} \right\}^{\frac{1}{4}} e^{i(\omega_0 t - ax)} \quad (32)$$

$$u_{12}(x, t) =$$

$$\left\{ -\frac{3(A_1^2 C^2 + 9AB)B \csc^2\left(2\sqrt{\frac{B}{A}}(v_0 t - vx)\right)}{C\left(3A\sqrt{\frac{B}{A}}\cot\left(2\sqrt{\frac{B}{A}}(v_0 t - vx)\right) - A_1 C\right)^2} \right\}^{\frac{1}{4}} e^{i(\omega_0 t - \omega x)} \tag{33}$$

选取参数 $\beta_2 = 1, \beta_3 = 1, \alpha_1 = 2, \gamma_1 = 1, \gamma_2 = 1, v_0 = 1, v = 1, \omega_0 = 1$ 进行图示。当 $A_1 = 1, \alpha_2 = 1, \omega = 2$ 时, 有 $A = \frac{5}{2}, B = -\frac{35}{9}, C = -\frac{5}{3}$, 可得方程解 (30)

为激波 (图 3(a)); 当 $A_1 = 3, \alpha_2 = -1, \omega = 2$ 时, 有 $A = \frac{5}{2}, B = -\frac{35}{9}, C = \frac{5}{3}$, 可得方程解 (31) 为亮-孤子 (图 3(b)); 当 $A_1 = 1, \alpha_2 = 1, \omega = 1$ 时, 有 $A = \frac{11}{6}, B = \frac{1}{18}, C = -\frac{5}{3}$, 相应可得方程解 (32) 和 (33) 的波形为周期波 (图 3(c)、(d))。

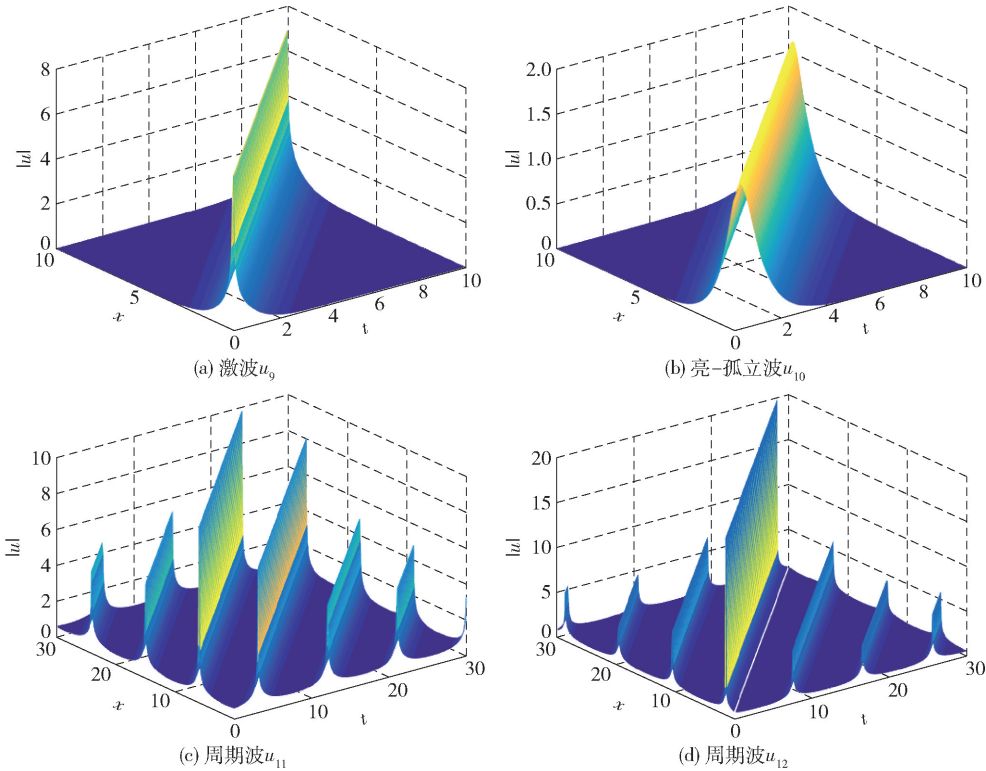


图 3 激波 u_9 , 亮-孤立波 u_{10} 和周期波 u_{11}, u_{12} 的三维波形图

Fig. 3 Three-dimensional waveforms of the shock wave u_9 , bright-soliton wave u_{10} and periodic waves u_{11}, u_{12}

通过比较可以发现以上 3 种方法存在内在关联性:

1) Riccati 方程-展开法的解表达最简, 不含自由参数, 倒数展开法的解有 2 个待定参数 A_0, A_1 转化为非 0 自由参数, Exp-展开法中有 1 个待定参数 A_1 转化为自由参数;

2) 倒数展开法中限定参数 $S_0 = 1$, 所得解即为 Exp-展开法求得的解;

3) Exp-展开法限定参数 $A_1 = 0$, 所得解即为 Riccati 方程-展开法求得的解。

3 结束语

本文结合文献[4-8]对系统(1)进行改进得到新方程(4)。该方程通过行波变换化简为二阶常微

分方程(6)。由于高次项 ϕ^5 和二阶导项 ϕ'' 在辅助函数法代入过程中关于 V 的次数变化幅度不同, 导致无法选出合适的 N 来达到次数平衡。为此, 引入变量代换 $\phi = Q^{1/4}$, 将方程(6)转化为方程(8), 运用多种辅助函数展开法—Riccati 方程-展开法、Riccati 方程-倒数展开法、Exp-展开法, 求解出几种类型的解析解, 如亮-孤立波、激波和周期波, 并且发现这 3 种方法求得的解存在一定的交集。

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Soliton solutions of a cubic-quintic nonlinear Schrödinger equation with third-order dispersion, a self-frequency shift and self-steepness term

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Abstract: According to the principle of homogeneous balance, the exact solutions of a cubic-quintic nonlinear Schrödinger equation with third-order dispersion, self-frequency shift and self-steepness terms have been obtained using the Riccati equation-expansion method, the Riccati equation-reciprocal expansion method, and the Exp – expansion method, including optical solitons, shock waves and periodic waves. We graphically illustrate its wave-form structure and compare the three methods.

Key words: nonlinear Schrödinger equation; Riccati equation-expansion method; Exp – expansion method; homogeneous balance principle; solitary wave

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