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中立型变时滞随机微分方程数值解的强收敛性

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摘 要: 讨论中立型变时滞随机微分方程改进的修正截断 Euler-Maruyama(EM)方法的 q 阶矩强收敛性,并得到收敛速度。结果表明此方法适用于高度非线性的漂移项和扩散项,且相较于隐式的修正截断 EM 方法计算量更小,适用范围更广。

关键词: 中立型变时滞随机微分方程; 改进的修正截断 Euler-Maruyama(EM)方法; 强收敛性

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引 言

中立型时滞随机微分方程是描述诸如医学、生态学、经济学、物理学等学科中众多现象的一种重要工具。然而,由于此类方程的复杂性,虽然其应用广泛,但是解析解却很难得到,因此研究其数值解以及数值解的收敛性很有必要。另一方面,过程的当前状态依赖于其过去的状态,但是这种依赖未必是常时滞的,从而使得考虑变时滞情形具有重要的实际意义。Mao 等[1-2]提出截断 Euler-Maruyama(EM)方法,并对该方法得到的数值解的收敛性和稳定性进行了研究。Milošević[3]和 Guo 等[4]分别研究了时滞随机微分方程和中立型时滞随机微分方程数值解的收敛性。Lan 等[5-7]提出了修正截断 EM 方法,并且得到典型随机微分方程和中立型常时滞随机微分方程的数值解的收敛速度和渐进稳定性。由于在变时滞情形下通常的修正截断方法会变为隐式格式,而相比于显式格式隐式格式通常计算量较大,因此本文将研究对应方程的显式格式(改进的修正截断 EM 方法)的收敛性及收敛速度。

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1 基本假设、定义及重要引理

设$(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},P)$是一个完备的概率空间,考虑中立型变时滞随机微分方程为

$$\mathrm{d}[x(t)-u(x(t-\delta(t)))]=f(x(t),x(t-\delta(t)))\mathrm{d}t+g(x(t),x(t-\delta(t)))\mathrm{d}B(t) \tag{1}$$

初值满足:$x_0=\xi=\{\xi(\theta),\theta\in[-\tau,0]\}\in C^b_{\mathscr{F}_0}([-\tau,0];R^n)$,其中 f,g,u $(f:R^n\times R^n\to R^n, g:R^n\times R^n\to R^n\otimes R^n, u:R^n\to R^n)$ 均为可测函数。为得到本文主要结论,提出以下假设。

H_1 对任意正整数 R,存在正常数 L_R,使得 $\forall x,\bar{x},y,\bar{y}\in R^n$,且 $|x|\vee|\bar{x}|\vee|y|\vee|\bar{y}|\leq R$,都有

$$|f(x,y)-f(\bar{x},\bar{y})|\vee|g(x,y)-g(\bar{x},\bar{y})|\leq L_R(|x-\bar{x}|+|y-\bar{y}|)$$

H_2 存在常数 $\eta\in(0,1)$,对 $\forall x,y\in R^n$,都有 $|u(x)-u(y)|\leq\eta|x-y|$,当 $u(0)=0$ 时,$|u(x)|\leq\eta|x|$。

H_3 函数 $\delta:R_+\to R_+$ 连续可微,且满足 $|\delta'(t)|<\bar{\delta}<1$。

H_4(Khasminskii-type 条件) 存在常数 $p\geq 2$ 和 $K>0$,使得对于 $\forall x,y\in R^n$,$a\in(0,1]$,有

$$\left\langle x-au\left(\frac{y}{a}\right),f(x,y)\right\rangle+\frac{p-1}{2}|g(x,y)|^2\leq K(1+|x|^2+|y|^2)$$

H_5 存在常数 C_ξ,对于 $p\geq 2$ 有

$$E\sup_{t,s\in[-\tau,0],|s-t|\le\Delta}|\xi(s)-\xi(t)|^p\le C_\xi\Delta^{\frac{p}{2}}$$

需注意的是由文献[3]可知,当 $H_1 \sim H_4$ 满足时,方程(1)的解 $x(t)$ 在初值 x_0 的条件下存在且唯一,并且对 $T>0$ 有 $\sup_{-\tau\le t\le T}E|x(t)|^p<\infty$。

定义停时 $\tau_R=\inf\{t\ge0,|x(t)|\ge R\}$,$\inf\phi=\infty$,则 $P(\tau_R\le T)\le\frac{C'}{R^p}$。

令 $\Delta\in(0,1)$ 为步长且 $\tau=m\Delta$,对于充分小的 $\Delta^*>0$,令 $h(\Delta)$ 是正的严格递减函数 $h:(0,\Delta^*]\to(0,\infty)$ 且满足

$$\lim_{\Delta\to0}h(\Delta)=\infty,\lim_{\Delta\to0}L^2_{h(\Delta)}\Delta=0 \tag{2}$$

由文献[6]知若给定 L_R,则函数 h 一定存在。

对任意的 $\Delta>0$,定义修正截断函数 f_Δ 为

$$f_\Delta(x,y)=\begin{cases}f(x,y), & |x|\vee|y|\le h(\Delta)\\ af(x_1,y_1), & |x|\vee|y|>h(\Delta)\end{cases}$$

式中,$a=\frac{|x|\vee|y|}{h(\Delta)}$,$x_1=\frac{x}{a}$,$y_1=\frac{y}{a}$。

下面定义改进的修正截断 EM 格式。令 $t_k=k\Delta$,$N=T/\Delta$,$X_k=\xi(t_k)$,$-m\le k\le0$,$X_{-m-1}:=X_{-m}$,若 $k=0,1,\cdots,N$,令

$$X_{k+1}=u(X_{k-I_k})+X_k-u(X_{k-1-I_{k-1}})+f_\Delta(X_k,X_{k-I_k})\Delta+g_\Delta(X_k,X_{k-I_k})\Delta B_k \tag{3}$$

式中,g_Δ 定义与 f_Δ 相同,其中 $I_k=\left[\frac{\delta(t_k)}{\Delta}\right]$,$k\ge0$,$I_{-1}:=I_0$,$[x]$为 x 的整数部分。

为定义连续格式,令

$$\bar{x}_\Delta(t)=\sum_{k=0}^{N-1}X_kI_{[t_k,t_{k+1})}(t) \tag{4}$$

$$\bar{y}_\Delta(t)=\sum_{k=0}^{N-1}X_{k-I_k}I_{[t_k,t_{k+1})}(t) \tag{5}$$

$$Z_k(t)=\left(1-\frac{t-t_k}{\Delta}\right)\bar{y}_\Delta(t_{k-1})+\frac{t-t_k}{\Delta}\bar{y}_\Delta(t_k),t\in[t_k,t_{k+1}) \tag{6}$$

$$\bar{z}_\Delta(t)=\sum_{k=0}^{N-1}Z_k(t)I_{[t_k,t_{k+1})}(t) \tag{7}$$

定义 $x_\Delta(t)=\xi(t)$,$t\in[-\tau,0]$,对 $\forall t\in[0,T]$,令

$$x_\Delta(t)=\xi(0)+u(\bar{z}_\Delta(t))-u(x_\Delta(-\Delta-I_{-1}\Delta))+\int_0^t f_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))\mathrm{d}s+\int_0^t g_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))\cdot\mathrm{d}B(s) \tag{8}$$

对 $\forall t\in[t_k,t_{k+1})$,上述格式也可写成如下形式

$$x_\Delta(t)=x_\Delta(t_k)+u(Z_k(t))-u(x_\Delta(t_{k-1}-I_{k-1}\cdot\Delta))+\int_{t_k}^t f_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))\mathrm{d}s+\int_{t_k}^t g_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))\mathrm{d}B(s) \tag{9}$$

容易验证 $x_\Delta(t_k)=\bar{x}_\Delta(t_k)=X_k$,$-m\le k\le N$。

引理 1 假设 H_1 成立,那么对于任意固定的 $\Delta>0$,有

$$|f_\Delta(x,y)-f_\Delta(\bar{x},\bar{y})|\vee|g_\Delta(x,y)-g_\Delta(\bar{x},\bar{y})|\le5L_{h(\Delta)}(|x-\bar{x}|+|y-\bar{y}|)$$

证明参见文献[7]中引理 3.1。

引理 2 假设 H_4成立,则有

$$\langle x-u(y),f_\Delta(x,y)\rangle+\frac{p-1}{2}|g_\Delta(x,y)|^2\le2K(1+|x|^2+|y|^2)$$

证明参见文献[7]中引理 3.2。

引理 3 对于 $\forall t\ge0$,有

$$E|\bar{z}_\Delta(t)|^p\le\sup_{-\tau\le s\le t}E|x_\Delta(s)|^p$$

证明 不妨设 $t\in[t_k,t_{k+1})$,由式(6)、(7)可知

$$E|\bar{z}_\Delta(t)|^p=E|Z_k(t)|^p=E\left|\bar{y}_\Delta(t_{k-1})+\frac{t-t_k}{\Delta}(\bar{y}_\Delta(t_k)-\bar{y}_\Delta(t_{k-1}))\right|^p\le\max\{E|\bar{y}_\Delta(t_{k-1})|^p,E|\bar{y}_\Delta(t_k)|^p\}\le\sup_{-\tau\le s\le t}E|x_\Delta(s)|^p$$

引理 4 假设 H_1、H_2、H_3、H_5成立,且 $\eta<\frac{1}{2}$,则存在常数 $C(p,T)>0$ 使得对任意的 $\Delta\in(0,\Delta^*)$,有

$$\sup_{k\le[\frac{t}{\Delta}]}E|x_\Delta(t_k)-x_\Delta(t_{k-1})|^p\le C(p,T)(L^p_{h(\Delta)}\Delta^{\frac{p}{2}}\sup_{s\le t}E|x_\Delta(s)|^p+\Delta^{\frac{p}{2}})$$

证明 由式(9)易知,

$$E|x_\Delta(t_k)-x_\Delta(t_{k-1})|^p=E|u(\bar{z}_\Delta(t_k))-u(\bar{z}_\Delta(t_{k-1}))+\int_{t_{k-1}}^{t_k}f_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))\mathrm{d}s+\int_{t_{k-1}}^{t_k}g_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))\mathrm{d}B(s)|^p\le(1+c)^{p-1}\eta^pE|\bar{z}_\Delta(t_k)-\bar{z}_\Delta(t_{k-1})|^p+\left(\frac{1+c}{c}\right)^{p-1}E\left[\left|\int_{t_{k-1}}^{t_k}f_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))\mathrm{d}s+\int_{t_{k-1}}^{t_k}g_\Delta(\bar{x}_\Delta(s),\bar{y}_\Delta(s))\mathrm{d}B(s)\right|^p\right]$$

又由式(6)、(7)可知

$$E|\bar{z}_\Delta(t_k)-\bar{z}_\Delta(t_{k-1})|^p=E|\bar{y}_\Delta(t_{k-1})-\bar{y}_\Delta(t_{k-2})|^p=E|X_{k-1-I_{k-1}}-X_{k-2-I_{k-2}}|^p$$

根据$[x]$的定义和假设 H_3可知

$$k-1-I_{k-1}-(k-2-I_{k-2})=1+I_{k-2}-I_{k-1}\le1+\left(\left[\frac{\delta(t_{k-2})-\delta(t_{k-1})}{\Delta}\right]+1\right)\le1+\left(\left[\frac{|\delta'(\theta)|\Delta}{\Delta}\right]+\right.$$

$1\Big)=2$

故

$$E|\bar z_\Delta(t_k)-\bar z_\Delta(t_{k-1})|^p\le 2^{p-1}\sup_{-m\le i\le k}E|x_\Delta(t_i)-x_\Delta(t_{i-1})|^p$$

又由引理 1,所以得到

$$\sup_{k\le[\frac{t}{\Delta}]}E|x_\Delta(t_k)-x_\Delta(t_{k-1})|^p\le[2(1+c)]^{p-1}\eta^p\sup_{i\le[\frac{t}{\Delta}]}E|x_\Delta(t_i)-x_\Delta(t_{i-1})|^p+\left(\frac{1+c}{c}\right)^{p-1}\sup_{k\le[\frac{t}{\Delta}]}E\left[\left|\int_{t_{k-1}}^{t_k}f_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\mathrm{d}s+\int_{t_{k-1}}^{t_k}g_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\mathrm{d}B(s)\right|^p\right]\le[2(1+c)]^{p-1}\eta^p\sup_{i\le[\frac{t}{\Delta}]}E|x_\Delta(t_i)-x_\Delta(t_i)|^p+\left(\frac{3(1+c)}{c}\right)^{p-1}(\Delta^p+\Delta^{\frac{p}{2}})L^p_{h(\Delta)}2\sup_{s\le t}E|x_\Delta(s)|^p+\left(\frac{3(1+c)}{c}\right)^{p-1}(|f(0,0)|^p\Delta^p+|g(0,0)|^p\Delta^{\frac{p}{2}})$$

由于 $\eta<\frac{1}{2}$,故可取 c 充分小使得 $[2(1+c)]^{p-1}\eta^p<1$,结论得证。

引理 5　设引理 4 中的条件均成立,则

$$E|x_\Delta(t)-\bar x_\Delta(t)|^p\le C(p,T)(L^p_{h(\Delta)}\Delta^{\frac{p}{2}}\sup_{s\le t}E|x_\Delta(s)|^p+\Delta^{\frac{p}{2}})$$

证明　不妨设 $t\in[t_k,t_{k+1})$。由式(9)得

$$E|x_\Delta(t)-\bar x_\Delta(t)|^p=E\left|u(Z_k(t))-u(x_\Delta(t_{k-1}-I_{k-1}\Delta))+\int_{t_k}^t f_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\mathrm{d}s+\int_{t_k}^t g_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\mathrm{d}B(s)\right|^p\le(1+c_0)^{p-1}\eta^pE|Z_k(t)-x_\Delta(t_{k-1}-I_{k-1}\Delta)|^p+\left[\frac{2(1+c_0)}{c_0}\right]^{p-1}\times E\left[\left|\int_{t_k}^t f_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\mathrm{d}s\right|^p+\left|\int_{t_k}^t g_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\mathrm{d}B(s)\right|^p\right]$$

注意到

$$|Z_k(t)-x_\Delta(t_{k-1}-I_{k-1}\Delta)|=\left|\left(1-\frac{t-t_k}{\Delta}\right)X_{k-1-I_{k-1}}+\frac{t-t_k}{\Delta}X_{k-I_k}-X_{k-1-I_{k-1}}\right|\le|X_{k-I_k}-X_{k-1-I_{k-1}}|$$

从而由引理 4 得

$$E|x_\Delta(t)-\bar x_\Delta(t)|^p\le C(p,T)(L^p_{h(\Delta)}\Delta^{\frac{p}{2}}\sup_{s\le t}E|x_\Delta(s)|^p+\Delta^{\frac{p}{2}})$$

根据引理 5 不难得到

$$E|\bar y_\Delta(t)-\bar z_\Delta(t)|^p\le C(p,T)(L^p_{h(\Delta)}\Delta^{\frac{p}{2}}\sup_{s\le t}E|x_\Delta(s)|^p+\Delta^{\frac{p}{2}})$$

引理 6　假设 $H_1\sim H_5$ 均成立,则

$$E|x_\Delta(t)-u(\bar z_\Delta(t))|^p\le C(p,T)\left(L^p_{h(\Delta)}\Delta^{\frac{p}{2}}\cdot\int_0^t\sup_{r\le s}E|x_\Delta(r)|^p\mathrm{d}s+\Delta^{\frac{p}{2}}\right)+C(p,\eta,T)$$

证明　由伊藤公式得

$$E|x_\Delta(t)-u(\bar z_\Delta(t))|^p\le E|\xi(0)-u(x_\Delta(-\Delta-I_{-1}\Delta))|^p+pE\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^{p-2}\Big[\bar x_\Delta(s)-\langle u(\bar y_\Delta(s)),f_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\rangle+\frac{p-1}{2}|g_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))|^2\Big]\mathrm{d}s+E\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^{p-2}\langle x_\Delta(s)-\bar x_\Delta(s),f_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\rangle\mathrm{d}s+E\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^{p-2}\langle u(\bar y_\Delta(s))-u(\bar z_\Delta(s)),f_\Delta(\bar x_\Delta(s),\bar y_\Delta(s))\rangle\mathrm{d}s=:I_1+I_2+I_3+I_4$$

下面分别估计 I_1、I_2、I_3、I_4 这 4 项。

$$I_1\le 2^{p-1}(E|\xi(0)|^p+\eta^pE|x_\Delta(-\Delta-I_{-1}\Delta)|^p)\le C(p,\eta)\sup_{-\tau\le s\le 0}E|\xi(s)|^p \tag{10}$$

由引理 2 和 Young 不等式得

$$I_2\le pKE\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^{p-2}\times(1+|\bar x_\Delta(s)|^2+|\bar y_\Delta(s)|^2)\mathrm{d}s\le C(p,\eta)\left(E\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^p\mathrm{d}s+\int_0^t\sup_{r\le s}E|x_\Delta(r)|^p\mathrm{d}s\right)+C(p,\eta,T) \tag{11}$$

由引理 1 和 Young 不等式得

$$I_3\le pE\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^{p-2}(|f(0,0)|^2+L^2_{h(\Delta)}|x_\Delta(s)-\bar x_\Delta(s)|^2+|\bar x_\Delta(s)|^2+|\bar y_\Delta(s)|^2)\mathrm{d}s\le C(p,\eta)E\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^p\mathrm{d}s+C(p,T)\left(L^p_{h(\Delta)}\Delta^{\frac{p}{2}}\int_0^t\sup_{r\le s}E|x_\Delta(r)|^p\mathrm{d}s+\Delta^{\frac{p}{2}}\right) \tag{12}$$

同理,由引理 5 和 Young 不等式可得

$$I_4\le p\eta E\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^{p-2}(|f(0,0)|^2+L^2_{h(\Delta)}|\bar y_\Delta(s)-\bar z_\Delta(s)|^2+|\bar x_\Delta(s)|^2+|\bar y_\Delta(s)|^2)\mathrm{d}s\le C(p,\eta)E\int_0^t|x_\Delta(s)-u(\bar z_\Delta(s))|^p\mathrm{d}s+C(p,T)$$

$$\left(L_{h(\Delta)}^{p}\Delta^{\frac{p}{2}}\int_{0}^{t}\sup_{r\leqslant s}E|x_{\Delta}(r)|^{p}\mathrm{d}s+\Delta^{\frac{p}{2}}\right) \tag{13}$$

由式(10)~(13)得

$$E|x_{\Delta}(t)-u(\bar{z}_{\Delta}(t))|^{p}\leqslant C(p,\eta)E\int_{0}^{t}|x_{\Delta}(s)-u(\bar{z}_{\Delta}(s))|^{p}\mathrm{d}s+C(p,T)\left(L_{h(\Delta)}^{p}\Delta^{\frac{p}{2}}\int_{0}^{t}\sup_{r\leqslant s}E|x_{\Delta}(r)|^{p}\mathrm{d}s+\Delta^{\frac{p}{2}}\right)+C(p,\eta,T)$$

再根据 Gronwall 引理[8]即可得证。

引理 7 假设 $H_1\sim H_5$ 均成立,且 $\eta<\frac{1}{2}$,$p>2$,则存在 $0<\Delta_0<\Delta^*$ 和 $C(p,T)>0$,使得对任意 $\Delta\in(0,\Delta_0)$,有 $\sup\limits_{0<\Delta<\Delta_0}\sup\limits_{0<t<T}E|x_{\Delta}(t)|^{p}\leqslant C<\infty$,$\forall T>0$。

定义停时 $\rho_{\Delta,R}=\inf\{t\geqslant0,|x_{\Delta}(t)|\geqslant R\}$,则 $P(\rho_{\Delta,R}\leqslant T)\leqslant\frac{C}{R^{p}}$。

证明

$$E|x_{\Delta}(t)|^{p}=E|x_{\Delta}(t)-u(z_{\Delta}(t))+u(z_{\Delta}(t))|^{p}\leqslant(1+c')^{p-1}E|x_{\Delta}(t)-u(z_{\Delta}(t))|^{p}+\left(\frac{1+c'}{c'}\right)^{p-1}\eta^{p}\cdot E|z_{\Delta}(t)|^{p}$$

当 c' 充分大时,$\left(\frac{1+c'}{c'}\right)^{p-1}\eta^{p}<1$,由引理 3、引理 6 和 Gronwall 引理得

$$\sup_{0<\Delta<\Delta_0}\sup_{0<t<T}E|x_{\Delta}(t)|^{p}\leqslant C<\infty,\ \forall T>0 \tag{14}$$

显然将式(14)中的 $x_{\Delta}(t)$ 换成 $x_{\Delta}(t\wedge\rho_{\Delta,R})$ 不等式仍成立。

再由 $R^{p}P(\rho_{\Delta,R}\leqslant T)\leqslant E(|x_{\Delta}(T\wedge\rho_{\Delta,R})|^{p})\leqslant C$ 可得 $P(\rho_{\Delta,R}\leqslant T)\leqslant\frac{C}{R^{p}}$。

引理 8 假设引理 7 中的假设均成立,则对 $\forall t\in[t_k,t_{k+1})$ 和充分小的 $\Delta(<1)$,$2<q<p$,有

$$E|\bar{y}_{\Delta}(t-\delta(t))-\bar{z}_{\Delta}(t)|^{q}\leqslant C(q,T)L_{h(\Delta)}^{q}\Delta^{\frac{q}{2}}$$

证明 显然 $t-\delta(t)\in[t_{k-1}-I_k\Delta,t_{k+1}-I_{k+1}\cdot\Delta)$,由 H_3,可得

$$t_{k+1}-I_{k+1}\Delta-(t_{k-1}-I_k\Delta)=2\Delta+(I_k\Delta-I_{k+1}\Delta)\leqslant3\Delta+[|\delta'(\theta)|]\Delta=3\Delta$$

故 $\bar{y}_{\Delta}(t-\delta(t))$ 可能取 X_{k-1-I_k}、X_{k-I_k}、X_{k+1-I_k},由引理 4 可得如下结果。

1)当 $\bar{y}_{\Delta}(t-\delta(t))=X_{k-1-I_k}$ 时,有

$$E|\bar{y}_{\Delta}(t-\delta(t))-\bar{z}_{\Delta}(t)|^{q}=E|X_{k-1-I_k}-\bar{z}_{\Delta}(t)|^{q}\leqslant2^{q-1}\left(1-\frac{t-t_k}{\Delta}\right)^{q}E|X_{k-1-I_k}-X_{k-1-I_{k-1}}|^{q}+2^{q-1}\cdot\left(\frac{t-t_k}{\Delta}\right)^{q}E|X_{k-1-I_k}-X_{k-I_k}|^{q}\leqslant2^{q-1}\sup_{-m\leqslant j\leqslant k}E|X_j-X_{j-1}|^{q}\leqslant C(q,T)L_{h(\Delta)}^{q}\Delta^{\frac{q}{2}}(\sup_{r\leqslant t}E|x_{\Delta}(r)|^{q}+\Delta^{\frac{q}{2}})$$

2)当 $\bar{y}_{\Delta}(t-\delta(t))=X_{k-I_k}$ 时,有

$$E|\bar{y}_{\Delta}(t-\delta(t))-\bar{z}_{\Delta}(t)|^{q}=E|X_{k-I_k}-\bar{z}_{\Delta}(t)|^{q}\leqslant E|X_{k-1-I_k}-X_{k-1-I_{k-1}}|^{q}\leqslant\sup_{-m\leqslant j\leqslant k}E|X_j-X_{j-1}|^{q}\leqslant C(q,T)L_{h(\Delta)}^{q}\Delta^{\frac{q}{2}}(\sup_{r\leqslant t}E|x_{\Delta}(r)|^{q}+\Delta^{\frac{q}{2}})$$

3)同理当 $\bar{y}_{\Delta}(t-\delta(t))=X_{k+1-I_k}$ 时,有

$$E|\bar{y}_{\Delta}(t-\delta(t))-\bar{z}_{\Delta}(t)|^{q}\leqslant3^{q-1}\sup_{-m\leqslant j\leqslant k}E|X_j-X_{j-1}|^{q}\leqslant C(q,T)L_{h(\Delta)}^{q}\Delta^{\frac{q}{2}}(\sup_{r\leqslant t}E|x_{\Delta}(r)|^{q}+\Delta^{\frac{q}{2}})$$

由引理 7,结论得证。

2 主要结果与证明

定理 1 设 $H_1\sim H_5$ 成立,则由式(8)定义的数值解 $x_{\Delta}(t)$ q 阶矩强收敛到精确解 $x(t)$,即对任意的 $p\geqslant2$,$q\in[2,p)$,有 $\sup\limits_{0\leqslant t\leqslant T}E|x_{\Delta}(t)-x(t)|^{q}\leqslant C_{q,T}L_{h(\Delta)}^{q}\cdot\Delta^{\frac{q}{2}}$。

证明 截断函数定义为 $F_R(x,y)=f_{h^{-1}(R)}(x,y)$ 和 $G_R(x,y)=g_{h^{-1}(R)}(x,y)$,取 Δ 充分小,易知对任意 $|x|\vee|y|\leqslant R\leqslant h(\Delta)$,$F_R(x,y)=f_{h^{-1}(R)}(x,y)=f(x,y)=f_{\Delta}(x,y)$,同理 $G_R(x,y)=g_{h^{-1}(R)}(x,y)=g(x,y)=g_{\Delta}(x,y)$。

设 $v(\theta)=\xi(\theta)$,$\theta\in[-\tau,0]$,当 $t\geqslant0$ 时考虑如下中立型变时滞随机微分方程。

$$\mathrm{d}[v(t)-u(v(t-\delta(t)))]=F_R(v(t),v(t-\delta(t)))\mathrm{d}t+G_R(v(t),v(t-\delta(t)))\mathrm{d}B(t) \tag{15}$$

对任意固定的 R,F_R、G_R 满足全局 Lipschitz 条件,故方程(15)有唯一的解 $v(t)$,$t\geqslant-\tau$,所以有

$$P(x(t\wedge\tau_R)=v(t\wedge\tau_R),\ \forall t\in[0,T])=1 \tag{16}$$

类似于方程(1)的数值解,可类似定义 $v_{\Delta}(t)$,$\bar{v}_{\Delta}(t)=\sum\limits_{k=0}^{N-1}V_kI_{[t_k,t_{k+1})}(t)$,$\bar{v}'_{\Delta}(t)=\sum\limits_{k=0}^{N-1}V_{k-I_k}\cdot I_{[t_k,t_{k+1})}(t)$,$\bar{v}'_k(t)=\left(1-\frac{t-t_k}{\Delta}\right)\bar{v}'_{\Delta}(t_{k-1})+\frac{t-t_k}{\Delta}\bar{v}'_{\Delta}(t_k)$,$t\in[t_k,t_{k+1})$,$\bar{v}'_{\Delta}(t)=\sum\limits_{k=0}^{N-1}\bar{v}'_k(t)I_{[t_k,t_{k+1})}(t)$。

由解的唯一性,方程(1)、(15)对应的数值解满足

$$P(x_{\Delta}(t\wedge\rho_{\Delta,R})=v_{\Delta}(t\wedge\rho_{\Delta,R}),\ \forall t\in[0,T])=1 \tag{17}$$

令 $l(t)=v(t)-u(v(t-\delta(t)))$,$l_{\Delta}(t)=v_{\Delta}(t)-$

$u(v'_\Delta(t))$，则

$$E|v(t)-v_\Delta(t)|^q\leqslant(1+c_1)^{q-1}E|l(t)-l_\Delta(t)|^q+\left(\frac{1+c_1}{c_1}\right)^{q-1}\eta^q E|v(t-\delta(t))-\bar{v}'_\Delta(t)|^q$$

$$E|l(t)-l_\Delta(t)|^q\leqslant 2^{q-1}T^{q-1}E\int_0^t|F_R(v(s),v(s-\delta(s)))-F_R(\bar{v}_\Delta(s),\bar{v}'_\Delta(s))|^q\mathrm{d}s+2^{q-1}T^{\frac{q}{2}-1}E\int_0^t|G_R(v(s),v(s-\delta(s)))-G_R(\bar{v}_\Delta(s),\bar{v}'_\Delta(s))|^q\mathrm{d}s\leqslant 2^{q-1}(T^{q-1}+T^{\frac{q}{2}-1})E\int_0^t L_R^q(|v(s)-\bar{v}_\Delta(s)|^q+|v(s-\delta(s))-\bar{v}'_\Delta(s)|^q)\mathrm{d}s\leqslant 2^{q-1}(T^{q-1}+T^{\frac{q}{2}-1})L_R^q\Big[E\int_0^t(|v(s)-v_\Delta(s)+v_\Delta(s)-\bar{v}_\Delta(s)|^q\mathrm{d}s+E\int_0^t|v(s-\delta(s))-v_\Delta(s-\delta(s))+v_\Delta(s-\delta(s))-\bar{v}_\Delta(s-\delta(s))+\bar{v}_\Delta(s-\delta(s))-\bar{v}'_\Delta(s)|^q\mathrm{d}s\Big]$$

根据引理 5 和引理 8 得

$$E|l(t)-l_\Delta(t)|^q\leqslant C(q,T)L_R^q E\int_0^t|v(s)-v_\Delta(s)|^q\mathrm{d}s+C(q,T)L_R^q\Delta^{\frac{q}{2}}(\sup_{r\leqslant t}E|x_\Delta(r)|^q+\Delta^{\frac{q}{2}})\tag{18}$$

同理可得

$$E|v(t-\delta(t))-v'_\Delta(t)|^q\leqslant\left(\frac{1+c_2}{c_2}\right)^{q-1}E|v(t-\delta(t))-v_\Delta(t-\delta(t))|^q+C(q,T)L_{h(\Delta)}^q\Delta^{\frac{q}{2}}(\sup_{r\leqslant t}E|x_\Delta(r)|^q+\Delta^{\frac{q}{2}})$$

取 c_1、c_2 充分大，使得 $\left(\frac{1+c_1}{c_1}\right)^{q-1}\eta^q\cdot\left(\frac{1+c_2}{c_2}\right)^{q-1}<1$，则

$$\sup_{s\leqslant t}E|v(s)-v_\Delta(s)|^q\leqslant\frac{C(q,T)L_R^q E\int_0^t|v(s)-v_\Delta(s)|^q\mathrm{d}s}{1-\left(\frac{1+c_1}{c_1}\right)^{q-1}\eta^q\left(\frac{1+c_2}{c_2}\right)^{q-1}}+\frac{C(q,T)L_R^q\Delta^{\frac{q}{2}}(\sup_{r\leqslant t}E|x_\Delta(r)|^q+\Delta^{\frac{q}{2}})}{1-\left(\frac{1+c_1}{c_1}\right)^{q-1}\eta^q\left(\frac{1+c_2}{c_2}\right)^{q-1}}\tag{19}$$

易知将 t 替换成 $t\wedge\theta_{\Delta,R}$，其中 $\theta_{\Delta,R}=\tau_R\wedge\theta_{\Delta,R}$，式(19)仍成立。类似于文献[6]中引理 4.1 的证明，由 Gronwall 引理得

$$\sup_{s\leqslant t}E|v(s\wedge\theta_{\Delta,R})-v_\Delta(s\wedge\theta_{\Delta,R})|^q\leqslant C_1L_R^q\cdot\exp(C_2L_R^qT)\Delta^{\frac{q}{2}}$$

其中，

$$C_1:=\frac{C(q,T)(\sup_{r\leqslant t}E|x_\Delta(r)|^q+\Delta^{\frac{q}{2}})}{1-\left(\frac{1+c_1}{c_1}\right)^{q-1}\eta^q\left(\frac{1+c_2}{c_2}\right)^{q-1}}$$

$$C_2=\frac{C(q,T)}{1-\left(\frac{1+c_1}{c_1}\right)^{q-1}\eta^q\left(\frac{1+c_2}{c_2}\right)^{q-1}}$$

令 $h(\Delta^*)=L^{-1}(L_R\exp(C_2L_R^qT/q))$，则 $\forall\Delta\in(0,\Delta^*)$，有

$$E|v(t\wedge\theta_{\Delta,R})-v_\Delta(t\wedge\theta_{\Delta,R})|^q\leqslant C_1L_R^q\cdot\exp(C_2L_R^qT)\Delta^{\frac{q}{2}}\leqslant C_1L_{h(\Delta)}^q\Delta^{\frac{q}{2}}$$

由式(16)、(17)以及停时的定义可得

$$E|x(t\wedge\theta_{\Delta,R})-x_\Delta(t\wedge\theta_{\Delta,R})|^q\leqslant C(q,T)L_{h(\Delta)}^q\Delta^{\frac{q}{2}}$$

再由标准的截断程序[1,5]和引理 7 即得

$$E|x(T)-x_\Delta(T)|^q\leqslant C(q,T)L_{h(\Delta)}^q\Delta^{\frac{q}{2}}$$

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Strong convergence of numerical solutions of neutral stochastic differential equations with time-dependent delay

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Abstract: Strong convergence of the modified truncated Euler -Maruyama method for stochastic differential equations with a time-dependent delay is discussed, and the convergence rate is obtained. This method can be applied to neutral stochastic differential delay equations (NSDDEs) with highly nonlinear drift and diffusion terms. Compared with the implicit modified truncated Euler -Maruyama method, the amount of calculation required is reduced and the application range is wider.

Key words: neutral stochastic differential equations with time-dependent delay; modified truncated Euler - Maruyama method; strong convergence

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